NOTE

ON LEXICOGRAPHIC PROBABILITY RELATIONS

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In this note I propose a definition for lexicographic probability relations and discuss the question, under what conditions is a probability relation lexicographic.

Key words: Lexicographic probability relation; tight probability relation.

1. Introduction

In this note I examine conditions under which a probability relation on a set of events is lexicographic. Chipman (1971) discussed the question of the minimal ordinal \( \alpha \) such that there exists an order-preserving function from an ordered set \( A \) (in his paper \( A \) is \( \mathbb{R} \) with any order on it) to \( \mathbb{R}^\alpha \) with the lexicographic order on it. Since there always exists an ordinal \( \beta \) such that there is an order-preserving function from \( A \) to the lexicographically ordered \( \mathbb{R}^\beta \) (see Chipman), one can define an order as lexicographic if there is no order-preserving function from \( A \) to \( \mathbb{R} \).

According to this definition, a countable set cannot be ordered lexicographically. Consider, however, the following example. Let \( Q \) be the set of all the finite unions of rational intervals \([a, b)\) in \([0, 2)\). \( Q \) is closed under complementation and finite intersections. Define on \( Q \) a relation \( \succeq \) as follows: for every \( A, B \in Q \), \( A \succeq B \) iff
\[
\mu(A \cap [0, 1)) > \mu(B \cap [0, 1)) \quad \text{or} \quad \mu(A \cap [0, 1)) = \mu(B \cap [0, 1)) \quad \text{and} \quad \mu(A \cap [1, 2)) \geq \mu(B \cap [1, 2))
\]
(\( \mu \) denotes the Lebesgue measure). \( Q \) is countable, but seems to be lexicographic. Indeed, there exists no probability function \( P \) on \( Q \) such that \( A \succeq B \) iff
\[
P(A) \geq P(B)
\]
(see Section 2).

A natural definition of lexicographic orders seems therefore as follows. An order \( R \) on a set \( X \) is lexicographic if there exist two orders \( R_1 \) and \( R_2 \) on \( X \) such that for every \( x, y \in X \), \( xR_1y \) iff \( xR_1y \), but not \( yR_1x \); or \( xR_1y, yR_1x, \) and \( xR_2y \). (\( R_1 \) and \( R_2 \) may themselves be lexicographic orders.)

This definition may encompass too much. For example, the decimal writing of real numbers induces on them an apparent lexicographic order. It seems, therefore, that if one wishes to avoid defining too simple orders as lexicographic, yet does not
want to restrict the set of lexicographic orders too narrowly, one must use some properties of the space itself, as, for example, its being a vector space (see Hausner, 1954), or a space of sets, as below.

In Section 2 I define lexicographic probability relations. In Section 3 I present conditions implying that a tight probability relation is lexicographic. The examples of Section 4 prove that these are sufficient, but not necessary conditions.

2. Definitions

Let \( S \) be a set of states of the world and let \( Q \subseteq 2^S \) be closed under complementation and finite intersections. Elements of \( Q \) are called events, denoted by \( A, B, C \) etc. Let \( \geq \) be a binary relation on \( Q \). \( A > B \) iff \( A \supseteq B \) but not \( B \supseteq A \), and \( A \sim B \) iff \( A \sim B \) and \( B \sim A \). \( \bar{A} \) denotes \( S \setminus A \).

**Definition.** A relation \( \geq \) on \( Q \subseteq 2^S \) is called a probability relation if

(a) \( \geq \) is complete and transitive;
(b) for every \( A \in Q \), \( A \geq \emptyset \) and \( S \geq \emptyset \);
(c) for every \( A, B, C \in Q \) such that \( A \cap C = B \cap C = \emptyset \), \( A \geq B \) iff \( A \cup C \geq B \cup C \).

**Proposition.** Let \( \geq \) be a probability relation on \( Q \) and let \( A, B, C, D \in Q \). If \( A \sim B \) and \( A \cap C = B \cap D = \emptyset \), then \( C \sim D \) iff \( A \cup C \geq B \cup D \).

**Proof.** \( A \sim B \Rightarrow (A \setminus B) \cup (C \setminus (B \cup D)) \geq (B \cup C) \setminus (A \cup D) \); hence,

\[
C \geq D \Rightarrow (B \cup C) \setminus (A \cup D) \geq (D \setminus C) \cup (B \setminus (A \cup C))
\]

\[
\Rightarrow (A \setminus B) \cup (B \setminus (A \cup C)) \geq (D \setminus C) \cup (B \setminus (A \cup C))
\]

\[
\Rightarrow A \cup C \geq B \cup D. \quad \square
\]

**Definition.** A probability relation \( \geq \) on \( Q \) is lexicographic if there exists an event \( A > \emptyset \) such that:

(1) \( \bar{A} \) includes an infinite number of non-equivalent events;
(2) for every \( B, C \in Q \), \( B \geq C \) iff \( B \cap \bar{A} > C \cap \bar{A} \), or \( B \cap \bar{A} \sim C \cap \bar{A} \) and \( C \cap \bar{A} \geq C \cap A \).

**Lemma 1.** A probability relation \( \geq \) on \( Q \) is lexicographic iff there exists \( A > \emptyset \) satisfying (1), such that

(3) for every \( B, C \subseteq \bar{A} \), \( B > C \Rightarrow B > C \cup A \).

**Proof.** (2) \( \Rightarrow \) (3): Let \( B, C \subseteq \bar{A} \) such that \( B > C \). \( B \cap \bar{A} = B > C = (C \cup A) \cap \bar{A} \), hence by (2), \( B > C \cup A \).

(3) \( \Rightarrow \) (2): Let \( B \geq C \). \( (B \cap \bar{A}) \cup A \geq B \geq C \geq C \setminus \bar{A} \); hence, \( B \cap \bar{A} \geq C \setminus \bar{A} \). Assume
that \( B \cap \bar{A} \sim C \cap \bar{A} \). \( (B \cap \bar{A}) \cup (A \cap B) = B \cap C = (C \cap \bar{A}) \cup (C \cap A) \). By the proposition, \( B \cap A \sim C \cap A \).

If \( B \cap \bar{A} \sim C \cap \bar{A} \), then \( B \cap A > (C \cap \bar{A}) \cup A \geq C \). If \( B \cap A \sim C \cap \bar{A} \) and \( B \cap A \geq C \cap A \), then by the proposition, \( B \geq C \). \( \square \)

In the next section I will use the following definitions and theorem, taken from Savage (1972):

**Definition.** Let \( Q \subseteq 2^S \) be a space of events and let \( \geq \) be a probability relation on it. \( P : Q \to [0, 1] \) is a probability function if \( P(\emptyset) = 0 \), \( P(S) = 1 \), and \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \). \( P \) strictly agrees with \( \geq \) iff \( A \geq B \Rightarrow P(A) \geq P(B) \). \( P \) almost agrees with \( \geq \) iff \( A \geq B \Rightarrow P(A) \geq P(B) \).

**Definition.** A probability relation \( \geq \) on \( Q \subseteq 2^S \) is fine if for every \( A > 0 \) there exists a partition of \( S \), with all its elements less probable than \( A \). A probability relation is tight if for every \( A > B \) there exists \( C > 0 \) such that \( B \cap C = \emptyset \) and \( A > B \cup C \).

**Theorem.** If \( \geq \) on \( Q \) is fine, then there is one and only one probability function almost agreeing with \( \geq \). If \( \geq \) is fine and tight, then this function strictly agrees with \( \geq \).

3. Lexicographic probability relations

Obviously, if \( \geq \) is lexicographic, then it cannot be represented by a strictly agreeing probability function. In this section I demonstrate conditions under which a probability relation which cannot be represented by a strictly agreeing probability function is lexicographic.

**Lemma 2.** Let \( \geq \) be a tight probability relation on \( Q \) and let \( P \) be an almost agreeing probability function on \( Q \). If there exists \( A > 0 \) satisfying (1), such that

4. \( P(A) = 0 \), and for every \( D \in Q \), \( P(D) = 0 \) implies \( A \geq D \), then \( \geq \) is lexicographic.

**Proof.** Let \( A > 0 \) satisfying (1) and (4), and let \( B, C \subseteq A \), \( B > C \). By the definition, \( B \cup A > C \cup A \). Since \( \geq \) is tight, there exists \( D > 0 \) such that \( (C \cup A) \cap D = \emptyset \) and \( B \cup A > C \cup A \cup D \). By (4), \( P(A \cup D) > 0 \). \( P(B) = P(B \cup A) \geq P(C \cup A \cup D) = P(C) + P(A \cup D) > P(C) = P(C \cup A) \); hence, \( B > C \cup A \). By Lemma 1, \( \geq \) is lexicographic. \( \square \)

This lemma cannot be reversed. A counter-example appears at the end of this note.
Definition. A set of events $Z = \{ A_i : i \in I \}$ is a chain if for every $i, j \in I$, $i \neq j$, it follows that $A_i \subset A_j$ or $A_j \subset A_i$. $Z$ is a maximal chain of events which have a certain property if for every event $B \in Z$ which has this property, $Z \cup \{ B \}$ is not such a chain.

Lemma 3. Let $\succeq$ be a probability relation on $Q$. If $P$ almost agrees with $\succeq$, then there exists a maximal chain of events with probability 0.

Proof. Denote by $\mathcal{I}$ the set of all the chains of events with probability 0. Elements of $\mathcal{I}$ are partially ordered by the inclusion relation. Let $\{ Z_j : j \in J \}$ be a chain of elements out of $\mathcal{I}$. Obviously, $\bigcup_{j \in J} Z_j$ is an upper boundary for this chain. By the Lemma of Zorn (see Halmos, 1960), there exist in $\mathcal{I}$ maximal elements; hence, there exists a maximal chain of events with probability 0.

Obviously, more than one such chain may exist. \[ \square \]

Theorem. Let $\succeq$ be a tight probability relation on $Q$. If in one of the maximal chains of events with probability 0 there exists a maximal event (containing all the events of this chain), then $\succeq$ is lexicographic.

Proof. Let $A$ be a maximal event of a maximal chain of events with probability 0. $\succeq$ is tight; hence, there exists no $B > A$ such that $P(B) = 0$. The theorem now follows from Lemma 2. \[ \square \]

4. Examples

The question arises whether there must always exist a maximal event with probability 0. If so, then by the theorem a tight probability relation which can be represented by an almost agreeing probability function (but not by a strictly agreeing probability function) must be lexicographic. The answer to this question is not necessarily affirmative.

Example. $S = [0, \infty)$, $Q$ is the set of all the finite unions of intervals $[\alpha, \beta)$ in $S$ ($\beta \leq \infty$). Define on $Q$ a probability relation $\succeq$ as follows. $A \succeq B$ iff $\mu(A) > \mu(B)$, or $\mu(A) = \mu(B) = \infty$ and $\mu(B) > \mu(A)$.

$\succeq$ is tight. Obviously, the only one probability function $P$ on $Q$ is given by

$$P(A) = \begin{cases} 0, & \mu(A) < \infty, \\ 1, & \mu(A) = \infty. \end{cases}$$

Of course, there is no maximal event with probability 0, and $\succeq$ is not lexicographic. Indeed, let $A \in Q$ such that $\mu(A) < \infty$ and let $B \subset \bar{A}$ such that $0 < \mu(B) \leq \mu(A)$. $B > \emptyset$, but $\emptyset \cup A \succeq B$. By Lemma 1, $\succeq$ is not lexicographic.
Denote this probability relation on \([0,\infty)\) by \(\succ^*\). The following example presents a tight and lexicographic probability relation which can be represented by an almost agreeing probability function, but does not include any maximal events with probability 0. In other words, Lemma 2 cannot be reversed.

**Example.** \(S = [-1,\infty)\), \(Q\) is the set of all the finite unions of intervals \([\alpha,\beta)\) in \(S\). \(A \succeq B\) iff \(A \cap [0,\infty) \succ^* B \cap [0,\infty)\), or \(A \cap [0,\infty) \sim^* B \cap [0,\infty)\) and \(\mu(A \cap [-1,0)) \geq \mu(B \cap [-1,0))\).

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**References**


