The Tail that Wags the Economy: Belief-Driven Business Cycles and Persistent Stagnation

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Abstract

The "great recession" was a deep downturn with long-lasting effects on credit markets, labor markets and output. We explore a simple explanation: This recession has been more persistent than others because it was perceived as an extremely unlikely event before 2007. Observing such an episode led all agents to re-assess macro risk, in particular, the probability of tail events. This change in beliefs endures long after the event itself has passed and through its effects on prices and choices, it produces long-lasting effects on investment, employment and output. To model this idea, we take a production economy and add agents who use standard econometrics tools to estimate the distribution of aggregate shocks. When they observe a new shock, they add that new piece of data to their data set and re-estimate the distribution from which it was drawn. Transitory shocks have persistent effects on beliefs because, once observed, the shocks stays forever in the agents’ data set. We feed a time-series of data on actual macro shocks into our model, let our agents re-estimate the distribution from which the data is drawn each period, and show that our belief revision mechanism can explain the 13% downward shift in trend output.

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1 Introduction

The “great recession” was a deep downturn with long-lasting effects on credit markets, labor markets and output. As Figure 1 shows, the financial crisis looks like a level shift in trend GDP. We explore a simple explanation: This recession has been more persistent than others because it was perceived as an extremely unlikely event. Observing crisis outcomes made us re-estimate macro risk. For example, in 2006, no one raised the possibility of financial panic. Today, the question of whether the financial crisis might repeat itself arises frequently and option prices continue to reflect heightened tail risk.

Importantly, these changes in beliefs can persist long after the event itself has passed. Since perceptions of tail risk affect prices and choices, this persistent change in beliefs has long-lasting output effects. To model this idea, we take a production economy and add agents who use standard econometric tools to estimate the distribution of aggregate shocks, in a non-parametric way. When they observe a new shock, they add that new piece of data to their data set and re-estimate the distribution from which it was drawn. Transitory shocks have persistent effects on beliefs because, once observed, the shocks stays forever in the agents’ data set. We feed a time-series of data on actual macro shocks for the US into our model, let our agents re-estimate the distribution from which the data is drawn each period, and show that our belief revision mechanism can explain the 13% downward shift in trend output.

Post-war recessions have typically had a distinct tough, followed by a sharp rebound toward trend outcome. In contrast, the great recession looks like a permanent level shift. For standard macro models or even belief-driven business cycle models to produce this kind of persistent deviation from trend, they need to feed in sufficiently persistent negative shocks. Not only is that approach at odds with the transitory 2008-09 shocks described by most data series, but also, by assuming persistence, such a theory provides no insight about why this recession looked so different. We propose a new type of belief-driven business cycle model where persistence is endogenous and state-dependent. The key difference is that our agents learn about the distribution of shocks, instead of a hidden or future state. We argue that fluctuations are persistent not because agents fear they are still in a “bad state,” but rather, because the experience of new extreme events permanently changes their assessment of risk. This view is quantitatively successful, supported by data, and consistent with popular narratives. In Figure 2, the skew index, which measures tail risk from option prices, shows a rise in the financial crisis, with no subsequent decline. Furthermore, the financial sector’s stagnation narrative emphasizes a change in “attitudes” or “confidence,” that we capture with belief changes, and the reductions in debt financing that result:

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1See Backus et al. (2015) for a formal analysis of propagation in business cycle models.
Figure 1: Real GDP in the U.S. and its trend, in logs. Blue line is a linear trend that fits data from 1950-2007. In 2014, real GDP was 0.13 below trend.

[Y]ears after U.S. investment bank Lehman Brothers collapsed, triggering a global financial crisis and shattering confidence worldwide, ... “The attitude toward risk is permanently reset.” A flight to safety on such a global scale is unprecedented since the end of World War II. The implications are huge: Shunning debt... can starve the global economy. (Huffington Post Oct.6, 2013)

To make the case that stagnation comes from belief revisions, our strategy is to take an off-the-shelf economic model of debt-financed firms, augmented with our belief-formation mechanism. We feed it with data from the post-war period through the financial crisis and compare its predictions to observed outcomes. Our contribution is not the economic model, but the belief-formation mechanism that generates endogenous persistence and offers a way to discipline belief-driven business cycle models by tying beliefs to data, without introducing free parameters.

Obviously, no one truly knows the distribution of economic shocks. That is a useful fiction economists use to discipline beliefs. Assuming that agents learn from data using standard econometric tools imposes just as much discipline. In addition, our approach has the advantage that it can produce persistent effects from transitory shocks; it can explain why measures of
disaster risk fluctuate even in periods when no disaster is observed; and it is simple to execute. While most learning models can explain a wide variety of phenomena with the right signal and state process, this non-parametric approach to belief updating ties beliefs firmly to observable data, without free parameters, and can be easily combined with a variety of sophisticated, quantitative macro models.

Since our contribution lies in the belief-formation mechanism, we begin in Section 2 by examining that mechanism in isolation, both theoretically and quantitatively. We feed in a data series on capital returns (because this is similar to what the driving force of the economic model will turn out to be) and estimate real-time kernel densities. This tells us how much agents who observed the financial crisis changed their beliefs about the distribution of capital returns. By simulating future capital returns, we can observe the nature and the persistence of this belief revision. The theory behind persistence is martingale property of beliefs: If an agent estimates probabilities that will predictably rise or fall in the future, they should revise their estimate today. Efficient estimates of fixed objects in a model are always martingales, meaning that each observation permanently alters the estimated probability of future events. Of course, if agents observe a sequence of moderate or positive shocks following the negative shock, the effect of the negative shock will diminish over time. But if the data observed in the future is consistent with the estimated distribution – mostly small shocks, with a few negative outliers – the left tail of the distribution will persist forever.

To gauge the macroeconomic effects of this belief mechanism, Section 3 embeds it in a
quantitative business cycle model. One approach would be to use a variant of a King et al. (1988) economy and allow agents to learn about technology shocks. Our mechanism can add persistence in such an environment. But such an approach has two problems: First, productivity did not fluctuate much in the crisis. Therefore, the effect on tail risk would be negligible. Second, this economy is not sensitive to tail risk. If we want to have any shot of explaining the great recession, we need to use an economic model where changes in tail risk create macro fluctuations. The state of the art in such models is Gourio (2012). The model features a continuum of firms that produce output with capital and labor. Following many papers in the financial crisis literature, Gourio (2012) uses capital quality shocks (shocks to the effective capital depreciation rate) as the driving force of the model because this series has large, but transitory fluctuations in the crisis. Before observing shocks, each firm chooses its labor and capital investment. Wages and investment can be financed with debt, which yields a tax advantage to the firm, but also subjects it to bankruptcy costs if it defaults. Our contribution to this model is to assume that firms do not know the true distribution of aggregate shocks. Each period, agents observe a new shock realization, add it to their data set, and use a normal kernel-density estimator to re-estimate the shock distribution. Instead of assuming persistent negative shocks to get stagnation, we use i.i.d. shocks and see how much persistence learning can generate endogenously.

We calibrate model parameters to match average leverage, including operating leverage, investment and default rates. The cost of issuing debt (the credit spread or risk premium) depends on the probability of default, which in turn, depends on the probability of adverse aggregate shocks. Thus, when the probability of a left tail event rises, the credit spread rises, debt issuance and real investment fall, and output declines.

Section 5 quantifies the effect of our belief revisions. To do this, we construct a time-series of capital quality shocks using historical data on replacement and market value of the non-financial capital stock from the Flow of Funds report. The increase in tail risk following the financial crisis triggers a cumulative drop in capital of 15-20% and in output of about 12%, with almost no rebound to trend. These predicted effects of the financial crises, are similar in magnitude to those in the data. Hall (2014) estimates that the U.S. capital stock and U.S. real GDP are each 13% lower than they would be if the economy had continued to grow at its pre-crisis rate of trend growth. Our model, which is driven by a temporary decline in capital quality and is calibrated to pre-crisis default and leverage data, matches the size of the output drop, without being calibrated to any GDP data. Hall also argues that the depressed rate of business capital formation was the single largest contributor to the persistent depressed output (often referred to as secular stagnation) in the post-crisis period. We use our model output to do the same decomposition as Hall does. We show that the symptoms of secular stagnation,
as seen in changes in investment and labor between 2008 and 2015, resemble those in the data. When we turn off the belief-revision mechanism and endow agents with the distribution implied by the full data set, the initial impact of the capital shock is similar, but all macro aggregates immediately start to rebound. Without belief updating, all the long-run effects disappear.

To better understand this result, we decompose the magnitude of the macro effects first quantitatively, then theoretically. In our numerical solutions, we turn off changes in the mean, then the variance, and finally remove debt. We find that each of these accounts for approximately one-third of the long-run output effect. Then we use steady state analysis to pinpoint the model features that make aggregates sensitive to probabilities.

Section 7 asks the model to incorporate other historical episodes and other forms of belief updating, such as discounting past data. It starts by asking: What if agents have knowledge of a long history of capital market shocks, including episodes like the great depression? It shows that doubling the sample size and including another large financial crisis only attenuates the long-run effect of the recent crisis by 30%. If we then incorporate a modest 1% annual discount rate for old data, the predictions are indistinguishable from the original model. These effects are just as persistent. Even discounting old data does not undermine the martingale property of beliefs. The next question is what But discounting does explain why even the infinite history of the world economy (albeit with varying data quality) might not teach us the true distribution of macro shocks. Next, we explore the persistence of smaller 1-standard-deviation shocks, like those that triggered the 2001 recession. We find fluctuations that are both smaller and less persistent. This exercise shows how, even if all belief changes are permanent, the persistence of output fluctuations depends on how rare the shock is. Events that are unlike previously-observed events produce the most persistent output fluctuations.

Not only does econometric belief formation help quantitatively by adding endogenous persistence to business cycle models, it also provides new economic intuition about the relationship between debt and real economic fragility. Investigating the process of statistical learning provides a new perspective on what it means for a price or quantity to be information insensitive. Typically, authors claim that debt is information-insensitive because its payoffs are flat throughout most of the state space. The one region where debt payoffs vary is in left-tail states that correspond to bankruptcy. Our learning mechanism reveals that beliefs in some regions of the state space are more sensitive to new information than others. In particular, beliefs about the probability of tail events are sensitive to new information because data in those regions is scarce. When data is scarce, new information strongly influences beliefs. Thus, debt is an instrument that is very sensitive to beliefs exactly in a region of the state space where beliefs are very sensitive to new information. Because of this effect, firms’ use of debt makes the economy more information-sensitive.
Taken together, our results suggest that the recovery from the great recession may have been slow simply because we learned that financial crises are still possible in the U.S. and this new knowledge permanently changed our assessment of macroeconomic risk.

**Comparison to the literature**  The production side of our model builds on existing work that traces out the macro consequences of an exogenous increase in disaster risk, e.g. Gourio (2012) and is similar in many respects to the existing theories of shocks to beliefs that drive business cycles\(^2\). Our approach based on re-estimating the distribution of shocks offers two key advantages. First, our belief shocks are not exogenous. Without discipline on the possible time-series of beliefs, many macroeconomic outcomes are rationalizable. Our agents’ beliefs are the outcome of a standard kernel-density estimation using actual macroeconomic data. The second advantage is that beliefs about fixed distributions are martingales, while beliefs about time-varying states are only persistent to the extent that one assumes the states or shocks are persistent. Our mechanism delivers persistent effects of transitory shock and offers an explanation for why some shocks are more persistent than others. This can help us understand why many recessions have rapid recoveries and yet, some do not.

A small number of uncertainty-based theories of business cycles also deliver persistent effects from transitory shocks. In Straub and Ulbricht (2013) and Van Nieuwerburgh and Veldkamp (2006), a negative shock to output raises uncertainty, which feeds back to lower output, which in turn creates more uncertainty. To get even more persistence, Fajgelbaum et al. (2014) combine this mechanism with an irreversible investment cost, a combination which can generate multiple steady-state investment levels. Getting large persistent fluctuations from this kind of hidden state model is a challenge for two reasons. One hurdle is that states that are far apart are easy to distinguish. So uncertainty about which state one is in will be short-lived, even with noisy data. So, hidden states must be relatively close together, which makes the economic impact of state uncertainty small. The second hurdle is that a 5% drop in output is a large economic fluctuation. But a 5% drop in signal precision has small effects. So these models need to build in mechanisms to amplify changes on both margins.

These uncertainty-based explanations leave two questions unanswered. First, why were credit markets hardest hit and most persistently impaired after the crisis? Second, why did the depressed level of economic activity continue long after the VIX and other measures of uncertainty had recovered? Our theory relies more on tail risk. Like uncertainty, tail risk is a moment of the perceived distribution of outcomes. But the value of debt is particularly sensitive

\(^2\)Papers on news driven business cycles include papers on news shocks, such as, Beaudry and Portier (2004), Lorenzoni (2009), Veldkamp and Wolfers (2007), papers on uncertainty shocks, such as Jaimovich and Rebelo (2006), Bloom et al. (2014), Nimark (2014) and papers on higher-order belief shocks, such as Angeletos and La’O (2013) or Huo and Takayama (2015).
to this moment. The skew index data in Figure 2 reveal that tail risk has lingered, making it a better candidate for explaining continued depressed output.

Our belief formation process is similar to the parameter learning models by Johannes et al. (2015), Cogley and Sargent (2005) and Orlik and Veldkamp (2014) and is advocated by Hansen (2007). However, none of these papers has a production economy or considers persistent shocks to output. In Pintus and Suda (2015), parameter learning in a production economy amplifies shocks to leverage. While they feed in persistent leverage shocks and explore amplification, we feed in large shocks to capital and explore endogenous persistence. In addition, our non-parametric approach allows us to incorporate beliefs about tail risk.

Our model draws on many popular theories of the great recession, such as Gertler et al. (2010), Gertler and Karadi (2011), Brunnermeier and Sannikov (2014) and Gourio (2012). Moriera and Savov (2015) is similar to our model in that agents learn and it changes their demand for shadow banking (debt) assets. But their agents learn about a hidden two-state Markov process, which has persistence built in. While this literature has taught us an enormous amount about the mechanisms that triggered declines in lending and output in the financial crisis, it also hard-wired in persistent shocks. Our model aims to complement these theories by describing a simple mechanism that is compatible with many existing frameworks, is easy to implement, and delivers persistence. Rather than substituting for these existing narratives about the mechanics of financial crisis, our belief-formation mechanism adds another layer to the story, by explaining why some shocks deliver more persistent responses than others.

Finally, our paper contributes to the literature on secular stagnation. Eggertsson and Mehrotra (2014) claim that secular stagnation prevails because of negative real interest rates and low effective demand. Other theories are surveyed by Hall (2015). While these alternatives may well be plausible, we offer a simple explanation for the post-crisis stagnation that can explain the propagation of business cycle shocks in normal times as well. The key ingredient that transforms transitory business cycle shocks into persistent fluctuations is allowing agents to estimate the shock distribution in real time, just like an econometrician would. The specifics of our production economy, our debt-financed firms, and the use of capital quality shocks all function to make the persistent changes in beliefs large enough to deliver a quantitatively plausible theory of secular stagnation. But, more broadly, our model describes a belief-formation mechanism that can add persistent effects of extreme shocks in many frameworks.

## 2 Estimating Beliefs

The contribution of this paper comes from explaining why tail risk fluctuates and why those fluctuations can create persistent responses to transitory shocks. In order to show that this is
economically important, the next section quantifies the economic effects with an off-the-shelf economic model. But instead of laying the whole model out at once, we begin by describing our novel piece of the model – how we model and estimate beliefs and why these beliefs add persistence to the underlying shock process. Once we explain our mechanism, then we embed this part of the model in an existing quantitative economic framework to assess how large an effect these belief changes have on macroeconomic aggregates.

No one knows the true distribution of shocks to the economy. We estimate such distributions. This implies that when new data arrives, we should re-estimate the distribution. The question is how to do this estimation. A common approach is to assume a normal distribution and estimate mean and variance. But, the normal distribution has a thin tail and is therefore not well-suited to measure changes in tail risk. We could choose some other functional form. But different functional assumptions will deliver different answers, all of which will therefore be suspect. Instead, we take a non-parametric approach and let the data speak about the shape of the distribution.

To lay out our main argument as simply as possible, consider an aggregate shock \( \phi_t \) whose true cumulative distribution \( G \) is unknown to agents in the economy. Importantly, however, they know that the shock \( \phi_t \) is i.i.d. The independence assumption ensures changes in beliefs are the only source of persistence in the long run. This distinguishes our results from those that exogenously assume persistence, by assuming that the \( \phi_t \) shocks are autocorrelated.

The common information set of our agents, denoted \( \mathcal{I}_t \), includes the history of all shocks \( \phi_t \) observed up to and including time-\( t \). At each \( t \), they use the available data to construct an estimate \( \hat{G}_t \) of the true distribution \( G \). Formally, at every date, agents construct the following normal kernel density estimator of the pdf \( g \):

\[
\hat{g}_t(\phi) = \frac{1}{n_t \kappa} \sum_{s=0}^{n_t-1} \Omega \left( \frac{\phi - \phi_{t-s}}{\kappa} \right)
\]

where \( \Omega (\cdot) \) is the standard normal density function, \( \kappa \) is the bandwidth parameter and \( n_t \) is the number of available observations of at date \( t \). We chose the normal kernel estimator with an optimal bandwidth because it is the most standard approach and places an enormous amount of discipline on the learning problem. We also studied a handful of other non-parametric and flexible parametric specifications, which yielded similar results. As new data arrives, agents add the new observation to their data set and update their estimates, generating a sequence of beliefs \( \{ \hat{G}_t \} \).

The key mechanism at work in the paper is the permanent effect on beliefs from realizations of the shock \( \phi_t \) that are purely transitory. The reason is that, conditional on time-\( t \) information \( (\mathcal{I}_t) \), the estimated probability \( \hat{G}_t(\phi) \) of any given shock \( \phi \) is a martingale. All innovations to
a martingale process are permanent. To understand where this permanent component comes from, simply apply the law of iterated expectations. The estimator \( \hat{\mathcal{G}}_t(\phi) \) is defined as the expectation of the true probability \( E[\mathcal{G}(\phi) | \mathcal{I}_t] \). Tomorrow’s estimator \( \hat{\mathcal{G}}_{t+1}(\phi) \) is similarly defined as tomorrow’s expectation of the true probability \( E[\mathcal{G}(\phi) | \mathcal{I}_{t+1}] \). Applying the law of iterated expectations, we know that \( E[E[\mathcal{G}(\phi) | \mathcal{I}_{t+1}] | \mathcal{I}_t] = E[\mathcal{G}(\phi) | \mathcal{I}_t] \). In other words, \( E[\hat{\mathcal{G}}_{t+1} | \mathcal{I}_t] = \hat{\mathcal{G}}_t \).

Intuitively, this tells us that if an agent believes today that tomorrow’s estimator will place a higher probability on event \( \phi \), then the agent should rationally increase today’s probability estimate of \( \phi \). Applying iterated expectations multiple times, we can show that \( E[\hat{\mathcal{G}}_{t+s} | \mathcal{I}_t] = \hat{\mathcal{G}}_t, \forall s \geq 1 \). This martingale property that all future values of the \( \hat{\mathcal{G}}_t \) process are expected to be equal to its current value implies that all changes in \( \hat{\mathcal{G}}_t \) are expected to be permanent.

We illustrate how this belief formation mechanism works by applying the estimation procedure described above to data on returns on business (i.e. non-residential) capital for the US economy in the post-WWII era. Our focus on capital returns is motivated by the striking effects of the financial crisis in 2008-09 on asset returns. Of course, capital returns are not exogenous variables. One would want them to emerge from some economic structure. In our economic model in the next section they do. We eventually introduce primitive shocks that are closely correlated with capital returns. Thus, by examining capital shocks directly, we offer a preview of the model dynamics to come.

We begin with the standard definition of returns:

\[
 r_t = \frac{d_t}{p_{t-1}} + \frac{p_t}{p_{t-1}} \tag{1}
\]

where the two components denote the dividend yield and capital gains in \( t \). For the former, we draw on the work of Gomme et al. (2011), who use data from the National Income and Product Accounts (NIPA) with careful adjustments for proprietors’ income, rents and taxes to construct a measure of returns to business capital in the US. We use the same data as in that paper and follow their methodology \(^3\) to generate an annual time series of dividend yield (defined as pre-tax capital income divided by the value of capital stock) for the period from 1948-2009. For the capital gain component, we use annual data on non-financial assets for US corporate entities. The Flow of Funds reports published by the Federal Reserve contain two such series - one evaluated at historical cost and the other at replacement cost or market value. We combine the two series, denoted \( NFA^{HC}_t \) and \( NFA^{RC}_t \) respectively and construct

\(^3\)We thank the authors for sharing their data and code with us.
the following measure of capital gains in period $t$:

$$\text{Nominal capital gains}_t = \frac{NFA_{t}^{RC}}{(1 - \delta)NFA_{t-1}^{RC} + \text{Invt}_t}$$

$$= \frac{NFA_{t}^{RC}}{(1 - \delta)NFA_{t-1}^{RC} + NFA_{t}^{HC} - (1 - \delta)NFA_{t-1}^{HC}}$$

Basically, the formulae in (3) adjust changes in the market/replace value series for new investments to isolate the part from revaluation effects. We then deflate this series using an aggregate price index to remove purely nominal changes and add it to the dividend yield series described earlier. The combined series is plotted in the left panel in Figure 3. It shows that realized returns during the financial crisis were significantly worse than any that were observed throughout the entire sample.

![Figure 3: Estimated Beliefs about Capital Returns, before and after the Financial Crisis](image)

The figure shows the estimated kernel density for 2007 and 2009 as well as the mean belief in 2039 (computed using data simulated from the estimated distribution in 2009.

When we estimate distributions using this data, the right panel of Figure 3 shows that these adverse realizations lead to an increase in the probability of a left-tail realization in the 2009 distribution, relative to the pre-crisis one. Crucially, even though these negative realizations were short-lived, this increase in left tail risk persists. To see how persistent they are, we ask the following question: What would be the mean belief in 2039 if the economy continued to be hit by shocks drawn from the estimated distribution in 2009? When we draw 30 years of realizations from the 2009 distribution and re-estimate beliefs using the combined actual and simulated data set, the average 2039 belief is the same at 2009 beliefs. Obviously, each simulated path gives rise to a different estimate, but averaging across paths yields the 2009 distribution. The simulation illustrates that, on average, concerns about tail events induced by
the financial crisis never go away.

Thus, every new shock to capital returns \((\phi_t)\), even ones that are transitory, have a permanent effect on beliefs. To assess the implications of these belief changes for macroeconomic outcomes, we need an economic model that maps shocks and beliefs into aggregate investment and hiring decision. The model laid out in the following section does just that - it is a set of specific assumptions that highlight the effect of tail risk on aggregate variables. However, we wish to point out that this flexible, non-parametric approach to belief formation is a simple tool that could be embedded in many other macroeconomic models to explore the potential for transitory shocks to generate persistent economic responses.

3 Economic Model

In order to understand how large a stagnation belief estimation can produce, we need to specify an economic environment. On this front, we do not innovate. Instead, we take an existing model, Gourio (2013), which in turn builds on Gertler and Karadi (2011) and Gourio (2012), and use it to quantify the size of the responses in capital, output and labor. In the Gourio (2013) model, firms finance investment and payroll expenses using a combination of debt and equity financing and are subject to aggregate and idiosyncratic shocks. Our innovation is to introduce real-time estimation of beliefs as a source of endogenous persistence in this framework.

Preferences and technology: An infinite horizon, discrete time economy has a representative household, with preferences over consumption and labor supply, following

\[
U_t = \left(1 - \beta \right) \left( C_t - \frac{\xi L_t^{1+\gamma}}{1+\gamma} \right)^{1-\psi} + \beta E_t \left( U_{t+1}^{1-\eta} \right)^{1-\psi} \right]\]

where \(\psi\) is the inverse of the intertemporal elasticity of substitution, \(\eta\) indexes risk-aversion and \(\gamma\) is inversely related to the elasticity of labor supply.

The economy is also populated by a unit measure of firms, indexed by \(i\) and owned by the representative household. Firms produce output with capital and labor, according to a standard Cobb-Douglas production function \(A k_{it}^{\alpha} l_{it}^{1-\alpha}\), where \(A\) is total factor productivity (TFP), which is the same for all firms and constant over time. Firms are subject to an aggregate shock to capital quality \(\phi_t\). A firm that enters the period with capital \(\hat{k}_{it}\) and is hit by a shock \(\phi_t\) has effective capital \(k_{it} = \phi_t \hat{k}_{it}\). These capital quality shocks \(\phi_t \sim G(\cdot)\) are the primary source of the model’s aggregate fluctuations.

Capital quality shocks are used in Gourio (2012), as well as in a number of recent papers
on financial frictions, crises and the Great Recession (e.g., Gertler et al. (2010), Brunnermeier and Sannikov (2014)). They were introduced in macro models by Gertler and Karadi (2011), who borrowed the idea from Merton (1973), in order to reconcile the high volatility in capital returns, with less-volatile macro series. These shocks scale up or down the effective capital stock. Of course, there are no shocks that regularly wipe out fractions of the capital or create it out of thin air. Instead, these shocks are a simple, yet imperfect way to capture the idea that a hotel built in Las Vegas may still be standing, but may deliver much less economic value after a financial crisis. Until we have a firmer understanding of why returns are so volatile, we need to use stand-in shocks like these if we want this kind of model to have any shot of speaking to both financial and macro data.

Firms are also subject to an idiosyncratic shock $v_{it}$. These shocks scale up and down the total resources available to each firm (before paying debt, equity or labor):

$$\Pi_{it} = v_{it} \left[ Ak_{it}^{1-\alpha} + (1-\delta)k_{it} \right]$$

where $\delta$ is the rate of capital depreciation. The shocks $v_{it}$ are i.i.d across time and firms and are drawn from a known distribution $F$. The mean of the idiosyncratic shock is normalized to be one: $\int v_{it} \, di = 1$

To keep our problem tractable, we follow most previous work on learning (Cogley and Sargent (2005), Piazzesi et al. (2015), Johannes et al. (2015)) in using anticipated utility (Kreps, 1998). At each date $t$, agents act as if the true distribution is $\hat{G}_t$. If we relax this assumption, it would not change how beliefs are updated. Nor would it affect the martingale properties of beliefs. Both are preference-independent. It would add a layer of uncertainty that would raise risk premia, as in Johannes et al. (2015). Since this extra uncertainty is not relevant to our main point about the persistence of real macro aggregates, and it adds considerable opacity and computational complexity, we suppress it.

**Labor, credit markets and default:** Firms hire labor in advance, i.e. before observing the realizations of aggregate and idiosyncratic shocks. Wages are non-contingent - in other words, workers are promised a non-contingent payment and face default risk.

Firms have access to a competitive non-contingent debt market, where lenders offer bond price (or equivalently, interest rate) schedules as a function of all relevant aggregate and idiosyncratic states, in the spirit of Eaton and Gersovitz (1981).

In order to characterize these schedules, we need to analyze the firm’s default decision. A firm enters the period with an obligation, $b_{it+1}$ to bondholders and a promise of $w_{it+1}l_{it+1}$ to

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$^4$This is a natural assumption - with a continuum of firms and a stationary shock process, firms can learn the complete distribution of any idiosyncratic shocks after one period.
its workers. The shocks are then realized and the firm (i.e. its shareholders) decide whether to repay their obligations or default. A firm that defaults makes no payments to equity holders. Formally, default is optimal for shareholders if

$$\Pi_{it+1} - b_{it+1} - w_{it+1}l_{it+1} + \Gamma_{t+1} < 0$$

where $$\Gamma_{t+1}$$ is the present value of continued operations (we characterize this object later in this section - specifically, we will show that, since idiosyncratic shocks are iid, this is the same for all firms and, in equilibrium, equal to 0). Thus, the default decision is a function of the resources available to the firm ($$\Pi_{it+1}$$) and the total obligations of the firm to both bondholders and workers ($$b_{it+1} + w_{it+1}l_{it+1} \equiv B_{it+1}$$). The former is a function of the capital and labor choices, as well as the realizations of shocks. Let $$r_{it+1} \in \{0, 1\}$$ denote the default policy of the firm.

In the event of default, the workers and bondholders take over the firm. The productive resources of a defaulting firm are sold to an identical new firm at a discounted price, equal to a fraction $$\theta < 1$$ of the value of the defaulting firm. The proceeds are distributed pro-rata among the creditors (both bondholders and unpaid workers). Note that the claims of both bondholders and workers have equal seniority.

Let $$q\left(\hat{k}_{it+1}, l_{it+1}, B_{it+1}, S_t\right)$$ denote the bond price schedule faced by a firm in period $$t$$. In other words, the firm receives $$q(\cdot)$$ in exchange for a promise to pay one unit of output at date $$t+1$$. Note that the bond price determination is made before the following period’s capital quality shocks are known. Therefore, the price depends on the amount of capital invested $$\hat{k}_{it+1}$$, but it cannot be made contingent on the effective capital that will be available for production $$k_{it+1}$$ or the profit shock $$v_{it+1}$$. The dependence on the other firm-level variables follows from our earlier discussion on the default decision. Formally,

$$q\left(\hat{k}_{it+1}, l_{it+1}, B_{it+1}, S_t\right) = \mathbb{E}_t M_{t+1} \left[ r_{it+1} + (1 - r_{it+1}) \frac{\theta \bar{V}(\Pi_{it+1}, S_{t+1})}{B_{it+1}} \right].$$

(6)

where $$\bar{V}(\Pi_{it+1}, S_{t+1})$$ is the value of the assets of the firm (to be characterized later) and $$M_{t+1}$$ is the stochastic discount factor of the representative household, which, given our Epstein-Zin specification takes the form

$$M_{t+1} = \left(\frac{dU_t}{dC_t}\right)^{-1} \frac{dU_t}{dC_{t+1}} = \beta \left[ E_t \left( U_{t+1} \right)^{1-\eta} \right]^{\frac{\eta-\psi}{1-\psi}} U_{t+1}^{\psi-\eta} \left( \frac{u(C_{t+1}, L_{t+1})}{u(C_t, L_t)} \right)^{-\psi}$$

(7)

Importantly, the bond price is a function of the aggregate state $$S_t$$, which includes the avail-

\(^{5}\text{Note also that this means that default does not destroy resources - the penalty is purely private.}\)
able history of aggregate shocks and outcomes. We will show later that $S_t$ can be summarized by three objects - aggregate resources available, denoted $\Pi_t$, the labor input $N_t$, (which is chosen in advance, i.e. in $t-1$) and the estimated distribution $\hat{G}_t$.

Debt is assumed to carry a tax advantage, which creates incentives for firms to borrow. A firm which issues debt at price $q_{it}$ and promises to repay $b_{it+1}$ in the following period, receives a total date-$t$ payment of $\chi q_{it} b_{it+1}$, where $\chi > 1$. This subsidy to debt issuance along with the cost of default introduces a trade-off in the firm’s capital structure decision, breaking the Modigliani-Miller theorem$^6$.

For a firm that does not default, the dividend payout is its total available resources times output shock, minus its payments to debt and labor, minus the cost of building next period’s capital stock (the undepreciated current capital stock is included in $\Pi_{it}$), plus the revenue earned from issuing new debt, including its tax subsidy:

$$d_{it} = \Pi_{it} - B_{it} - \hat{k}_{it+1} + \chi q_{it} b_{it+1}$$ (8)

Importantly, we do not restrict dividends to be positive, with negative dividends interpreted as (costless) raising of equity. Thus, firms are not financially constrained, ruling out another potential source of persistence.

Workers, who are also members of the representative family, evaluate their wage claims using the stochastic discount factor, $M_{t+1}$. This implies that the present value of a promise of wage $w_{it+1}$ is given by

$$w_{it+1} \mathbb{E}_t M_{t+1} \left[ r_{it+1} + \frac{(1 - r_{it+1})}{B_{it+1}} \frac{\theta \hat{V}(\Pi_{it+1}, S_{t+1})}{\hat{G}_{it+1}} \right] = w_{it+1} q_{it}$$

where the expectation is taken over aggregate and idiosyncratic shocks. From the household’s problem, we can derive the following optimality condition for labor supply:

$$w_{it+1} q_{it} \frac{dU_t}{dC_t} = \frac{dU_t}{dL_{t+1}}$$

$$w_{it+1} q_{it} = \left( \frac{dU_t}{dC_t} \right)^{-1} \frac{dU_t}{dL_{t+1}}$$

$$\equiv W_t$$ (9)

In other words, the expected value of wages, weighted by the economy-wide stochastic discount factor $M_{t+1}$ is the same for all firms and is equal to the marginal rate of substitution.

---

$^6$The subsidy is assumed to be paid by a government that finances it through a lumpsum tax on the representative household.
of the representative household. The wage promise, \( w_{it+1} \), must offer workers compensation for default risk. Since the risk is identical for bonds and wage payments, this risk adjustment involves simply multiplying the promised wage by the equilibrium bond price. In other words, the workers are essentially paid through bonds.

**Belief formation** No agent in the model knows the true distribution of capital quality shocks \( G \). The common information set of our agents, denoted \( \mathcal{I}_t \), includes the history of all shocks \( \phi_t \) observed up to and including time-\( t \). As in section 2, agents use all \( t \)-available data \( \mathcal{I}_t \) to construct an estimated distribution \( \hat{G}_t \), using (2), a normal kernel density estimator. As new data arrives, agents add the new observation to their data set and update their estimates, generating a sequence of beliefs \( \{ \hat{G}_t \} \).

**Timing, value functions and equilibrium:** The timing of events in each period \( t \) is as follows:

1. Firms enter the period with a capital stock \( \hat{k}_{it} \), labor \( l_{it} \), outstanding debt \( b_{it} \), and a wage obligation \( w_{it} l_{it} \).
2. The aggregate capital quality shock \( \phi_t \) and the firm-specific profit shock \( v_{it} \) are realized. Production takes place.
3. The firm decides whether to default or repay \( (r_{it} \in \{0, 1\}) \) its bond and labor claims.
4. The firm makes capital \( \hat{k}_{it+1} \), debt \( b_{it+1} \) choices for the following period, along with wage/employment contracts \( w_{it+1} \) and \( l_{it+1} \). Workers commit to next-period labor supply \( l_{it+1} \). Note that all these choices are made concurrently.

**Value functions:** In recursive form, the problem of the firm is

\[
V(\Pi_{it}, B_{it}, S_t) = \max \left[ 0, \max_{d_{it}, k_{it+1}, b_{it+1}, w_{it+1}, l_{it+1}} d_{it} + \mathbb{E}_t M_{t+1} V(\Pi_{it+1}, B_{it+1}, S_{t+1}) \right] \quad (10)
\]
subject to

\begin{align*}
\text{Dividends:} & \quad d_{it} \leq \Pi_{it} - B_{it} - \hat{k}_{it+1} + \chi q_{it} b_{it+1} \\
\text{Discounted wages:} & \quad \mathcal{W}_t \leq w_{it+1} q \left( \hat{k}_{it+1}, l_{it+1}, B_{it+1}, S_t \right) \\
\text{Future obligations:} & \quad B_{it+1} = b_{it+1} + w_{it+1} l_{it+1} \\
\text{Resources:} & \quad \Pi_{it+1} = v_{it+1} A(\phi_{t+1} \hat{k}_{it+1})^{\alpha} l_{it+1}^{1-\alpha} + (1 - \delta) \phi_{t+1} \hat{k}_{it+1} \\
\text{Bond price:} & \quad q \left( \hat{k}_{it+1}, l_{it+1}, B_{it+1}, S_t \right) = \mathbb{E}_t M_{t+1} \left[ r_{it+1} + (1 - r_{it+1}) \frac{\theta \tilde{V}_{it+1}}{B_{it+1}} \right]
\end{align*}

The first max operator in (10) captures the firm’s option to default if the value of the firm is negative. The expectation is taken over the idiosyncratic and aggregate shocks, taking the estimated aggregate shock distribution in \( S_t \) as given. Finally, the value of the assets of a defaulting firm \( \tilde{V} (\Pi_{it}, S_t) \) is simply the value of a firm with no external obligations, i.e. \( V (\Pi_{it}, 0, S_t) = \tilde{V} (\Pi_{it}, S_t) \).

The two aggregate objects that are relevant for the firm’s problem are the wage rate \( \mathcal{W}_t \) and the stochastic discount factor, \( M_{t+1} \). These depend on aggregate consumption and labor, implying that the aggregate state \( S_t \) can be summarized by \( (\Pi_t, L_t, \hat{G}_t) \) where \( \Pi_t \equiv AK_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t \) is the aggregate resources availa.

For a given \( \hat{G}_t \), an anticipated utility recursive equilibrium is a set of (i) functions for aggregate consumption and labor that maximize (4) subject to a budget constraint, (ii) firm value functions and associated policy functions that solve (10), taking the bond price and wage functions (6), and (9) and the stochastic discount factor (7) as given. (iii) aggregate consumption and labor are consistent with individual choices.

4 Solving the model

We now characterize the equilibrium and explore how tail events and the subsequent changes in beliefs affect the persistence and level of macro and financial outcomes. We present only the key equations here and relegate the detailed derivations to Appendix A.

We first note that, for a given aggregate shock \( \phi_t \), we can represent the optimal default policy as a threshold rule in the idiosyncratic output shock \( v_{it} \),

\[
r_{it} = \begin{cases} 
0 & \text{if } v_{it} < \psi(S_t) \\
1 & \text{if } v_{it} \geq \psi(S_t)
\end{cases}
\]

In the Appendix, we show that the optimality condition with respect to capital can be expressed
as follows:

\[ 1 + \chi \mathcal{W}_t \frac{l_{t+1}}{k_{t+1}} = \mathbb{E}[M_{t+1}R_{t+1}^k] + (\chi - 1)\text{lev}_{t+1}q_t - (1 - \theta)\mathbb{E}[M_{t+1}R_{t+1}^k h(v)] \quad (11) \]

where \( R_{t+1}^k = A\phi_{t+1}^{\alpha} \left( \frac{\hat{k}_{t+1}}{l_{t+1}} \right)^{\alpha-1} + (1 - \delta)\phi_{t+1} \)

\[ h(v) \equiv \int_{-\infty}^{v} v f(v) dv \]

The term \( R_{t+1}^k \) is the *ex-post* per-unit, pre-wage return on capital, \( \phi_{t+1} \), while \( h(v) \) is the expected value of the idiosyncratic shock under default.

The first term on the right hand side of (11) is the usual expected direct return from investing, weighted by the stochastic discount factor. This would be the only term in a world without debt (or in the problem of a planner maximizing the representative household’s utility). The other two terms are related to debt. The second term reflects the indirect benefit to investing arising from the tax advantage of debt - for each unit of capital, the firm raises \( \text{lev}_{t+1}q_t \) from the bond market and earns a subsidy of \( \chi - 1 \) on the proceeds. The last term is the cost of this strategy - the deadweight loss associated with default, equal to a fraction \( 1 - \theta \) of available resources.

All 3 components of the return to investing are influenced by beliefs and therefore, by changes to beliefs. When agents see large negative realizations, they update their beliefs about the likelihood of similar outcomes in the future. This increase in tail risk drives down the expected direct returns. It also raises risk premia (acting through the stochastic discount factor \( M_{t+1} \)). Both these effects serve to lower the first term, reducing incentives to invest. The effects on the debt-related terms in (11) is more complicated. However, our numerical results will show that the change in beliefs post-2009 reduces the net returns to capital from those two terms as well. Intuitively, tail risk lowers the effective tax subsidy (through lower bond prices) and raises the deadweight costs of default.

A few more lines of algebra yield the following representation of equation (11):

\[ 1 + \chi \mathcal{W}_t \frac{l_{t+1}}{k_{t+1}} = \mathbb{E}[M_{t+1}R_{t+1}^k J^k(v)] \quad (12) \]

where \( J^k(v) = 1 + h(v)(\theta \chi - 1) + v(1 - F(v))(\chi - 1) \)

where \( J^k(v) \) reflects the net effect of distortions induced by debt and can, therefore, be inter-

---

7Since all firms are identical, they make symmetric choices and accordingly, we suppress the \( i \) subscript.
interpreted as a wedge, which distorts equilibrium capital choice away from the choices of a planner. In the absence of debt (e.g. if \( \chi = 1 \)), \( J^k(v) = 1 \), reducing (11) to a standard Euler equation.

The optimality condition for labor looks quite similar. Just like with capital, firms equate the marginal cost of an additional unit of labor, namely \( W_t \), with the expected marginal product of labor, adjusted for the effect of additional promised wages on the cost of default:

\[
\chi W_t = E_t \left[ M_{t+1} (1 - \alpha) A \phi_t^{\alpha} \left( \frac{\hat{k}_{t+1}}{l_{t+1}} \right)^{\alpha} J^l(v) \right]
\]  

where \( J^l(v) = 1 + h(v) (\theta \chi - 1) - v^2 f(v) \chi (\theta - 1) \)

Finally, the firm’s optimality condition with respect to leverage

\[
(1 - \theta) E_t [M_{t+1} v f(v)] = \left( \frac{\chi - 1}{\chi} \right) E_t [M_{t+1} (1 - F(v))] \]  

The left hand side is the marginal cost of increasing leverage - it raises the expected losses from the default penalty (a fraction \( (1 - \theta) \) of the firm’s value). The right hand side is the marginal benefit - the tax advantage times the value of debt issued.

The three optimality conditions, (12) – (14), along with those from the household side - in particular, the labor supply condition (9) - characterize the equilibrium of this economy and can be solved numerically.

For computational tractability, we make a simplifying assumption - instead of letting firms choose bond obligations period-by-period, we assume that they follow a simple rule and target a constant leverage (defined as the ratio of total obligations to capital). This is equivalent to replacing (14) with

\[
\frac{B_{it+1}}{\hat{k}_{it+1}} = l e v^{\text{Target}}
\]

5 Measurement and Calibration

One of the key strengths of our belief-driven theory is that, by assuming that agents form beliefs as an econometrician would, we allow the data to discipline beliefs. In this section, we parameterize the model to match key features of the US economy. We then subject the model economy to the realized time series of capital quality shocks from US post-war data and evaluate the predictions for aggregates that we did not calibrate to, such as investment, output and consumption.
5.1 Measuring capital quality shocks

The next step is to construct a time series of \( \{ \phi_t \} \). We use the two series on non-financial assets in the US economy from the Flow of Funds reports that we used in Section 2. Recall that one series was evaluated at historical cost while the other at replacement cost or market value. In the language of our model, the latter series can be interpreted as effective capital, or \( K_t \). Letting \( X_{t-1} \) denote investment in period \( t-1 \) and \( P_t \) the nominal price of capital goods in \( t \), we can formally map these objects into their model counterparts as follows:

\[
P^k_t K_t = NFA_{t}^{RC}
\]
\[
P^k_{t-1} K_t = (1 - \delta) NFA_{t-1}^{RC} + P^k_{t-1} X_{t-1}
\]
\[
= (1 - \delta) NFA_{t-1}^{RC} + NFA_{t}^{HC} - (1 - \delta) NFA_{t-1}^{HC}
\]

where

\[
NFA_{t}^{RC} = \text{Replacement cost of non-financial assets}
\]
\[
NFA_{t}^{HC} = \text{Historical cost of non-financial assets}
\]

To adjust for changes in the nominal price \( P_t \), we use the price index for non-residential investment from the National Income and Product Accounts (denoted \( PINDX_t \))\(^8\). This allows us to recover the quality shock \( \phi_t \):

\[
\phi_t = \frac{K_t}{\hat{K}_t} = \left( \frac{P^k_t K_t}{P^k_{t-1} K_{t-1}} \right) \left( \frac{P^k_{t-1}}{P^k_t} \right) \tag{15}
\]
\[
= \left[ \frac{NFA_{t}^{RC}}{(1 - \delta) NFA_{t-1}^{RC} + NFA_{t}^{HC} - (1 - \delta) NFA_{t-1}^{HC}} \right] \left( \frac{PINDX_{t}^{k}}{PINDX_{t-1}^{k}} \right) \tag{16}
\]

where the second line replaces \( \left( \frac{P^k_{t-1}}{P^k_t} \right) \) with \( \left( \frac{PINDX_{t}^{k}}{PINDX_{t-1}^{k}} \right) \).

Using the measurement equation (16), we construct an annual time series for capital quality shocks for the US economy over the last few decades. The left panel of Figure 4 plots the resulting series. For most of the sample period, the shock realizations are in a relatively tight range around 1, but at the onset of the recent Great Recession, we saw two large adverse realizations: 0.93 in 2008 and 0.84 in 2009. To put these numbers in context, the mean and standard deviation of the series from 1950-2007 were 1 and 0.03 respectively.

We then apply our kernel density estimation procedure to this time series to construct a

\(^8\)Our results are robust to alternative measures of nominal price changes, e.g. computed from the price index for GDP or Personal Consumption Expenditure.
sequence of beliefs. In other words, for each $t$, we construct $\{\hat{g}_t\}$ using the available time series until that point. The resulting estimates for two dates - 2007 and 2009 - are shown in the right panel of Figure 4. They show that the great recession induced a significant increase in the perceived tail risk. The density function for 2007 implies almost zero mass below 0.95, while the one for 2009 attach a non-trivial probability to significantly worse realizations.

5.2 Calibration

A period is interpreted as a year and the discount factor $\beta$ is set to 0.9. The share of capital in the production, $\alpha$, is set to 0.40. The recovery rate upon default, $\theta$, is set to 0.70, following Gourio (2013). The distribution for the idiosyncratic shocks, $v_{it}$, is assumed to be lognormal, i.e. $v_{it} \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$ with $\hat{\sigma}^2$ chosen to target a default rate of 0.02. The labor supply parameter, $\gamma$, is set to 0.5, in line with Midrigan and Philippon (2011), corresponding to a Frisch elasticity of 2. The labor disutility parameter $\zeta$ and the TFP term in production are normalized to 1.

For the parameters governing risk aversion and intertemporal elasticity of substitution, we use standard values from the asset pricing literature and set $\psi = 0.5$ (or equivalently, an IES of 2) and $\eta = 10$. The leverage target is 0.70, obtained by adding the wage bill (approximately 0.2 of the steady state capital stock) to financial leverage (the ratio of external debt to capital, which stands at roughly 0.5 in US data - from Gourio (2013)). Since leverage is exogenous, the tax advantage $\chi$ is a free parameter. We set it to a baseline value of 1.06 and verified numerically that our results are not particularly sensitive to this choice.

Table 1 summarizes all parameter choices.

---

9 This is in line with the target in Khan et al. (2014), though a bit higher than the one in Gourio (2013). We verified that our quantitative results are not sensitive to this target.

10 See discussion in Gourio (2013).
Table 1: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\beta$</td>
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<td>Discount factor</td>
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<tr>
<td>$\eta$</td>
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<td>1/Intertemporal elasticity of subst.</td>
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<td>$\gamma$</td>
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<td>Depreciation rate</td>
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<tr>
<td>$A$</td>
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<td>TFP</td>
</tr>
<tr>
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<td>Tax advantage of debt</td>
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<td>$\theta$</td>
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<td>Recovery rate</td>
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<tr>
<td>$lev^{Target}$</td>
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<td>Leverage ratio</td>
</tr>
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</table>

6 Quantitative Results

Our main goal in this section is to quantify the size and persistence of the macroeconomic response to a large but transitory shock $\phi_t$ and explore the role of belief updating, debt and risk aversion to this response. With this goal in mind, we perform the following experiment using historical data on $\phi_t$ realizations from 1950-2009, measured using the strategy outlined in Section 5. We begin by estimating $\hat{G}_{2007}$ using the data through 2007. Then, starting from the steady state associated with this estimated distribution\(^{11}\), we subject the model economy to two adverse realizations - 0.93 and 0.84, which correspond to the shocks that we observed in 2008 and 2009. This leads to a revised estimate for the distribution, $\hat{G}_{2009}$. At this point, we could keep adding 2010-14 data, re-estimating each period (we do this in the Appendix). We could even simulate data beyond 2014 by drawing from the estimated distribution. But we know that beliefs are martingales. The average distribution estimated on draws from $\hat{G}_{2009}$ is $\hat{G}_{2009}$. Therefore, we hold beliefs to be $\hat{G}_{2009}$ and assume that the shock realizations from 2010 on are equal to their average value. This exercise isolates the persistent response to an isolated shock to beliefs, without mixing up that endogenous response with the effect of additional exogenous shocks.\(^{12}\)

The resulting impulse response functions are shown in Figure 5. The top left panel shows the...
time path for $\phi_t$ (as deviations from its average value). The remaining panels show the behavior of output, capital and employment over time. They show that the negative realizations in 2008 and 2009 and the resulting belief revisions induce a prolonged stagnation, with the economy trending towards a new, lower (stochastic) steady state. Output in this new steady state is 13% lower than the associated one under $\hat{G}_{2007}$. The corresponding drops in capital and labor are about 19% and 9% respectively. Thus, even though the $\phi_t$ shocks were transitory, they were so large that the resulting change in beliefs permanently reduces economic activity.

![Graphs showing economic variables over time](image-url)

**Figure 5:** Large negative shocks create extremely persistent responses in output, investment and labor. Solid line shows the change in aggregates (relative to the stochastic steady state associated with $\hat{G}_{2007}$). The circles show de-trended US data for the period 2009-2014. Dashed line (no learning) is an identical model where agents believe that shocks are drawn from the distribution estimated on the full sample of data and never revise those beliefs.

Figure 5 also plots the actual data on the output, capital stock and labor (in deviations from their respective pre-2007 trends) for the US economy.\(^{13}\) As the graph shows, the model’s predictions for the drop in output line up remarkably well with the data. The predicted path for capital and labor are also similar to the observed patterns though there are some differences. The former exhibits a sharper drop than the observed drop in capital input - which is not par-

\(^{13}\)We use data on output, capital and labor input from Fernald (2014). Each series is adjusted for growth in working age population and then detrended using a log-linear trend estimated using data from 1950-2007.
particularly surprising, given that our model abstracted from adjustment costs and other frictions that could induce a more sluggish response of capital. Similarly, the model’s predictions for labor underpredict the actual change in employment. In the data, employment dropped sharply in 2008-09, almost contemporaneously with the negative shocks and then recovered slowly. In the model, however, the drop occurs later, but that is largely due to the assumption that labor is chosen in advance. Bringing the model closer to the data along these dimensions is no doubt important and will require a richer model with additional features and frictions, but Figure 5 demonstrates the quantitative potential of learning in a standard business cycle setting\textsuperscript{14}.

To demonstrate the central role of learning, Figure 5 also plots the impulse responses (the dashed lines) for an otherwise identical economy where agents are assumed to know the final distribution \( \hat{G}_{2009} \) throughout from the very beginning and so, do not revise their beliefs. This corresponds to a standard rational expectations approach, where agents are assumed to know the true distribution of shocks hitting the economy and the econometrician estimates this distribution using all the available data. The impulse-response functions are computed as for the baseline case (the solid lines): each economy is assumed to be at its stochastic steady state in 2007 and is subjected to the same sequence of shocks - two large negative ones in 2008 and 2009 and the average level subsequently.

They show that, in the absence of belief revisions, the negative shock prompts firms to increase investment to replenish the lost effective capital. While the curvature in the utility function moderates the speed of this transition to an extent, the overall pattern of a steady recovery back to the original steady state is clear\textsuperscript{15}. This shows that learning is central to the model’s ability to generate long-lived reductions in economic activity.

6.1 Decomposing the long-run effects

We now perform a series of counterfactual experiments with our calibrated model to isolate the various forces that combine to deliver the results presented in Figure 5. The first analyzes the contribution of changes in the first moment of the distribution (i.e. in the average \( \phi \)), while the second quantifies the role of risk aversion by analyzing a version with quasi-linear preferences. The third experiment shows the interaction between learning and debt by comparing our results to an economy with no leverage. Table 2 summarizes our main conclusions. It shows that all 3 components - learning, debt and risk aversion play a significant role in generating persistent stagnation. Removing any of these elements would eliminate between one-third and one-half of

\textsuperscript{14}In the Appendix, we show that including the effect of shock realizations post-2009 does not materially change this finding.

\textsuperscript{15}Since the no-learning economy is endowed with the same end-of-sample beliefs as the learning model, they both ultimately converge to the same \textit{levels}. But they start at different steady states (normalized to 0 for each series).
2014 Long run

<table>
<thead>
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<th>Data</th>
<th>Benchmark model</th>
<th>Counterfactuals</th>
<th>No learning</th>
<th>Constant mean</th>
<th>No debt</th>
<th>No risk aversion</th>
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</tr>
</tbody>
</table>

Table 2: Change in GDP (relative to 2007 steady state).

our long-run effects.

6.1.1 Role of changes in mean $\phi$

Figure 6: Higher moments account for almost half of the long-run effects. Solid line (Baseline) is the model in section 2, where beliefs are updated according to (2). Dashed line (Constant mean) is an identical model where the estimation distribution $\hat{G}_t$ is re-scaled so that $E_t(\phi_t) = 1$. Zero is the initial steady state level in each economy.

Here, we decompose the total effect of belief revisions into a component attributable to changes in the mean (or average $\phi$) and the remaining attributable to changes in higher mo-
ments. To do this, we adjust the estimated distribution in 2009 so that $E_{2009}(\phi_t) = E_{2007}(\phi_t)$. The dashed lines in Figure 6 show the impulse response functions under this modification. They show that even with the mean change taken out, the long-run fall in GDP is about 6%, just under half of the total effect in our baseline case (the solid lines). In other words, while the effects coming from mean changes are significant, the contribution of higher moment changes is just as important in generating the persistent decline in economic activity from belief revisions.

The change in the mean $E_t\phi_t$ between 2007 and 2009 is relatively modest, about 0.4%, but, as the graph shows, its effect on long-run is about 7%. To dig a little deeper into why long-run outcomes are very sensitive to $\phi$, we turn to a special case - a deterministic version of our economy without debt. In steady state, the level of capital is governed by this intertemporal Euler equation:

$$
\ln k_{ss} = \text{Const.} + \left(\frac{1 + \gamma}{\gamma} \frac{\alpha}{1 - \alpha}\right) \ln \phi_{ss} - \left(\frac{1}{1 - \alpha} \frac{\alpha + \gamma}{\gamma}\right) \ln \left(\frac{1}{\beta} - (1 - \delta) \phi_{ss}\right)
$$

$$
\Rightarrow \frac{d \ln k_{ss}}{d \ln \phi_{ss}} = \left(\frac{1 + \gamma}{\gamma} \frac{\alpha}{1 - \alpha}\right) + \left(\frac{1}{1 - \alpha} \frac{\alpha + \gamma}{\gamma}\right) \frac{(1 - \delta) \phi_{ss}}{1/\beta - (1 - \delta) \phi_{ss}}
$$

Under our parameterization,

$$
\frac{1 + \gamma}{\gamma} \frac{\alpha}{1 - \alpha} = 2, \quad \frac{1}{1 - \alpha} \frac{\alpha + \gamma}{\gamma} = 3, \quad \left(\frac{(1 - \delta) \phi_{ss}}{1/\beta - (1 - \delta) \phi_{ss}}\right)_{\phi_{ss}=1} = 7.5
$$

which implies

$$
\frac{d \ln k_{ss}}{d \ln \phi_{ss}} = 2 + 3(7.5) = 24.5
$$

This simple calculation shows the source of the high sensitivity - the fact that capital quality shock affects not just the current return component but also the portion that comes from the undepreciated stock. In other words, it has a bigger effect on the total return to capital than say, total factor productivity. This combined with the high elasticity of steady state capital lies behind the significant effects of the change in the mean $\phi$.

### 6.1.2 Role of risk aversion

Finally, we investigate the implications of risk aversion by comparing our results to an otherwise identical economy with quasilinear preferences. Formally, we set $\psi = \eta = 0$, so the utility function of the representative household reduces to $C_t - \xi \frac{L_{t+\gamma}^\delta}{(1+\gamma)\delta}$. This eliminates the desire for consumption smoothing and risk premia. While these elements are no doubt important in reality, suppressing them allows us to see how much of the persistent drop in investment and
hiring comes from changes in the perceived distribution of returns to investing.

Figure 7 presents the time path for aggregate variables in this version, both with and without learning. As we would expect, the absence of curvature in consumption means that the economy transitions immediately to the new steady state. However, belief revisions still have substantial, permanent effects on the level of economic activity. For example, they lead to a drop in steady state output of about 7%. With risk aversion, this drop is almost doubled. This is because now capital (and labor) have to earn a risk premium, which changes with beliefs, and is particularly sensitive to tail risk. This further damps firms’ incentives to invest (and hire) after observing a large negative shock. The graph indicates that this effect is quite strong and accounts for almost half of the long-run drop in output.

6.1.3 Role of debt

Next, we examine the interaction between learning and debt by comparing our results to an identical economy where all investment is financed through equity (beliefs are updated over time, exactly as in our baseline model). Formally, we set the tax advantage parameter $\chi$ to
Figure 8: **Debt amplifies the effects of belief revisions on output, investment and labor.** Solid line (debt) is the model in section 2. Dashed line (no debt) is an identical model with $\chi = 0$, where firms choose zero debt. Zero is the steady state level in each economy.

1 and the leverage target to 0. This implies that $J^k(\nu) = J^l(\nu) = 1$, i.e. the debt-related distortions in capital and labor choice disappear.

Figure 8 plots the time path for aggregate variables for this variant of our model, along with our baseline version from Figure 5. The graph shows that the effects of belief revisions are smaller in the absence of debt though still significant - they lead to a 9% long-run reduction in output (compared to 13% in the version with debt). Thus, the presence of defaultable debt amplifies the effects of changes in tail risk and contributes almost a third of the long run macroeconomic response.

**Large vs small shocks:** Recall from Figure 4 that the adverse shocks observed during the 2008-’09 period were exceptionally large - the shock in 2009, for example, was almost 5 standard deviations below the mean. To better understand the effect of such a large shock, we conduct a counter-factual simulation and subject the model to a much smaller shock - 1 standard deviation below the mean, which is roughly in line with what we observed during the 2001-’02 recession. The results are shown in Figure 9. The $1\sigma$ shock has a much smaller effect on impact, but perhaps more interestingly, the effects of aggregate outcomes are transitory,
Figure 9: **Small shocks generate transitory effects.** Solid line (2008 recession) is the model with shocks equal to those observed during 2008-09. Dashed line (2001 recession) is the counter-factual simulation with a $1\sigma$ shock.

Figure 10: **Small shocks imply negligible belief revisions.** The solid blue line in both panels shows the estimated density in 2007, while the dashed lines show the new estimate after the 2008-09 shocks (left panel) and a counter-factual $1\sigma$ shock (the right panel).

unlike those arising from the 2008-'09 shock. Both shocks induce permanent belief revisions but the magnitude of these changes are much smaller under the $1\sigma$ shock. As a result, the new stochastic steady state is not that much different from the starting point, causing the economy to return, albeit slowly, to more or less the same level of economic activity as before the shock.
With the 2008-09 shock, however, the change in beliefs (and through them, on aggregates) is quite dramatic, leading to very different long run outcomes. This experiment reveals the ability of our learning model to rationalize why some recessions (like the Great Recession) can look very different from earlier episodes\textsuperscript{16}.

The size of shocks also matters for the amplification from debt. The attractiveness of debt (and therefore, the incentives to borrow) is affected disproportionately by perceived tail risk - and since adding larger shocks to the information set changes beliefs further out in the tail, they are amplified to a much greater extent by debt. In contrast, smaller shocks induce belief changes closer to the mean and therefore, do not interact with debt in a significant fashion. This is demonstrated in Figure 6.1.3. To generate the picture, we subjected our model economy calibrated to the 2007 beliefs with shocks ranging in size from 1\(\sigma\) to 5\(\sigma\). The left panel shows the change in beliefs at the corresponding point in the support of the distribution - for example, adding a new observation that was 5 standard deviations below the mean the changes the pdf at that point by 10\(\times\)10\(^{-3}\). The right panel plots the corresponding effects on GDP in the long-run (i.e. at the new stochastic steady state). It shows that debt adds significant non-linearity to the response of the economy to large shocks - the responsiveness to small shocks is pretty much almost the same as without debt, but larger shocks see significant amplification.

![Figure 11: Debt amplifies belief revisions from large shocks.](image)

The left panel shows the change in the estimated pdf from the addition of shocks of various sizes to the information set in 2007. The right panel shows the corresponding changes in long run GDP both with (solid line) and without debt (dashed line).

\textsuperscript{16}Note that this is not simply a statement about large vs small shocks - what matters is not the size of the shock \textit{per se} but the effect it has on beliefs. For example, large shocks also may have transitory effects in an economy where such shocks have been observed very frequently.
7 The Great Depression

Of course, the financial crisis was not the first time in history that the U.S. economy has experienced large adverse economic shocks. It is certainly possible that knowledge of previous shocks, particularly the Great Depression, affects how beliefs changed in response to the recent financial crisis. Since our simulations start in 1950, the experience of the 1930s is not in our agents’ information set. The primary constraint here is data availability - the data that we use to construct a time series of $\phi_t$ is available only for the post-WW II period.

But, apart from data availability, extending the history of data used for learning also raises another issue - how do agents incorporate data from the very far past into their belief formation process? If such data are treated on par with more recent observations, then, as the observed data series lengthens, beliefs converge, and eventually, new data ceases to affect beliefs. Our previous analysis suggests that beliefs about tail events are likely to converge more slowly than those elsewhere in the distribution, but even so, convergence is ultimately inevitable. However, if agents discount or underweight past data, such convergence will not occur. The notion of discounting is used extensively in the learning literature\textsuperscript{17} and is consistent with econometric learning in a setting where there is a possibility of an unobserved regime shift. It also arises in models of experiential learning with overlapping generations.

In this section, we present some numerical exercises intended to explore the implications of including a longer data sample on the effects of the financial crisis. We will show that under reasonable assumptions about the data series pre-1950 and the rate at which past data are discounted, the belief changes induced by the 2008-09 experience has a large, persistent effect on economic activity.

As mentioned earlier, lack of data prevents us from directly measuring $\phi_t$ before 1950. Our first attempt to deal with the lack of data was to come up with an indirect measure. To this end, we projected the measured $\phi_t$ series post-1950 on a number of variables and used the estimated coefficients to impute values for $\phi_t$ pre-1950. However, this did not quite work, despite a fairly comprehensive search\textsuperscript{18}. Our second, and admittedly cruder, attempt was to use the post-WW II sample to construct scenarios for what the pre-WW II sample might have looked like. Given that our goal is not to explain what happened during the Great Depression years, but rather, to understand how having more data affects learning today, this seems a reasonable place to start. We begin with the simplest exercise - replicate the 1950-2009 sample, i.e. assume that $\phi_t$

\textsuperscript{17}See, for example, Sargent (2001), Cho et al. (2002) and Evans and Honkapohja (2001).

\textsuperscript{18}We used a range of macro and asset pricing variables for this exercise - including GDP, unemployment, S&P returns and the Case-Shiller index of home prices. We also experimented with lead-lag structures to overcome the somewhat contemporaneous correlation with our measured $\phi_t$ series. Across specifications, the resulting projections for 1929-1930 showed only modestly adverse realizations.
realizations for the period from 1890-1949 were identical to those in 1950-2009. We make one adjustment to the sequence of shocks - for 1929-30, we use the 2008-09 realizations. In other words, the Great Depression saw adverse shocks that were comparable to the recent financial crisis\textsuperscript{19}. We then repeat our analysis under the assumption that agents are endowed with this expanded data series (starting from 1890). Now, when the financial crisis hits, the effect on beliefs is moderated by two factors - one, the larger data sample, which reduces the influence of new observations and two, the fact that agents had ‘seen’ a similar episode before.

![Figure 12: Extending the data sample does not materially change the effect of belief revisions.](image)

Each panel plots the response of GDP to the 2008-09 shocks under a hypothetical information set, starting from 1890. To fill in the data for the period 1890-1949, we use the observed time series from 1950-2009, with the realizations for 1929-30 set equal to those in 2008-09. In the last panel, the realizations for 1929-30 are assumed to be twice as bad as the Great Recession. The parameter $\lambda$ indexes the extent to which older observations are discounted. $\lambda = 1$ represents no discounting.

To capture discounting, we modify our kernel estimation procedure to allow for weights. Observation from $s$ periods earlier are assigned a weight $\lambda^s$, where $\lambda \leq 1$ is a parameter. When this is set to 1, there is no discounting of past observations\textsuperscript{20}.

The first panel of Figure 12 shows the impulse response without discounting (i.e. with $\lambda = 1$). It shows a slow increase in output following the crisis and some attenuation of the long-run effect, but the changes relative to the baseline case in Figure 5 are modest. When

\textsuperscript{19}Later, we perform another experiment where we relax this assumption and make the $\phi_t$ shocks during the Great Depression years twice as worse as the 2008-09 observations.

\textsuperscript{20}When $\lambda < 1$, eventually, the financial crisis observation will be discounted as well. Without additional assumptions about the true process of shocks, this will introduce non-stationarities into our analysis. Since our primary goal here is to show the effect of longer data samples, we avoid this complication and maintain the assumption of anticipated utility, i.e. agents are myopic with respect to belief changes in the future. In other words, they believe that the discounted kernel estimate is the true (time-invariant) distribution of shocks hitting the economy. Our simulations are done under this assumption and, as before, present results for mean path of beliefs. As a result, despite discounting, the belief revision is permanent.
older data is discounted by 1% ($\lambda = 0.99$, the center panel), this attenuation almost completely disappears and the impulse responses lie right on top of our baseline estimates\textsuperscript{21}.

Of course, we are probably understating the true magnitude of the shocks experienced during the Great Depression by setting them equal to those seen in 2008-09. Therefore, we redo the analysis under the assumption that the shocks in 1929-30 were twice as bad as those in 2008-09. This implies that $(\phi_{1929}, \phi_{1930}) = (0.86, 0.70)$. Note that these are very large shocks - they are respectively 5 and 10 standard deviations below the mean and taken together, imply an erosion of almost 50% in the stock of effective capital for the entire US economy. The results from this version (with the discounting parameter $\lambda$ set to 0.99) are presented in the third panel of Figure 12. Again, it shows some attenuation of the long-run effects, but as in the other cases, this effect is quite modest.

In sum, these results suggest that expanding the information set by adding more data does not drastically alter our main conclusions, especially once we allow for the possibility of discounting.

8 Conclusion

No one knows the true distribution of shocks to the economy. Economists typically assume that agents in their models do know this distribution as a way to discipline beliefs. But assuming that agents do the same kind of real-time estimation that an econometrician would do is equally disciplined and more plausible. For many applications, assuming full knowledge has little effect on outcomes and offers tractability. But for outcomes that are sensitive to tail probabilities, the difference between knowing these probabilities and estimating them with real-time data can be large. The estimation error can be volatile and can introduce new, persistent dynamics into a model with otherwise transitory shocks. The essence of the persistence mechanism is this: Once observed, a shock (a piece of data) stays in one’s data set forever and therefore permanently affects belief formation.

When firms finance investments with debt, they make investment and output sensitive to tail risk. Debt is an asset whose payoffs are flat throughout most of the state space, but very sensitive to the state for left-tail, default events. Therefore, the cost of debt depends precisely on the probabilities of a tail event, which are hardest to estimate and whose estimates fluctuate greatly. When debt (leverage) is low, the economy is not very sensitive to tail risk, and economic shocks will be more transitory. The combination of high debt levels and a shock that

\textsuperscript{21}If we keep increasing the discount, the decline in long-run GDP becomes bigger. Intuitively, with high enough discounting, the weight of recent observations increases beyond the level in the undiscounted, reduced sample used for our baseline analysis. For example, with $\lambda = 0.98$, GDP drops by about 16% in the new steady state.
is a negative outlier makes tail risk surge, investment fall and depresses output in a persistent way.

When we quantify this mechanism and use capital price and quantity data to directly estimate beliefs, our model’s predictions resemble observed macro outcomes in the wake of the great recession. These results suggest that perhaps persistent stagnation arose because, after seeing how fragile our financial sector is, market participants will never think about tail risk in the same way again.
References


Appendix

A  Optimality conditions from firm’s problem

The firm’s optimization problem is

\[ V(k_{it}, l_{it}, w_{it}, b_{it}, S_t) = \max \{0, \ \Pi_{it} - B_{it} + \Gamma_{it}\} \]

\[ \Gamma_{it} = \max_{\hat{k}_{it+1}, \ b_{it+1}, w_{it+1}, l_{it+1}} -\hat{k}_{it+1} + \chi q_{it+1} + EM_{t+1} V(k_{it+1}, l_{it+1}, w_{it+1}, b_{it+1}, S_{t+1}) \]

\[ \Pi_{it} = v_{it} \left( A(\phi_t k_{it})^\alpha l_{it+1}^{1-\alpha} + (1 - \delta) \phi_t k_{it} \right) \]

\[ B_{it+1} = b_{it+1} + w_{it+1} l_{it+1} \]

\[ q\left(\hat{k}_{it+1}, l_{it+1}, B_{it+1}, S_{t+1}\right) = \mathbb{E}_t M_{t+1} \left[ r_{t+1} + (1 - r_{t+1}) \frac{\theta V(k_{it}, l_{it}, 0, 0, S_t)}{B_{it+1}} \right] \]

\[ w_{it+1} q = W_t \]

\[ r_{it+1} = \begin{cases} 0 & \text{if } v_{it} < \underline{v}(S_t) \\ 1 & \text{if } v_{it} \geq \underline{v}(S_t) \end{cases} \]

We begin by noting that the firm’s problem can be expressed in terms of choosing leverage, defined as \(lev_{it+1} \equiv \frac{B_{it+1}}{k_{it+1}}\), and the labor-capital ratio, \(\frac{l_{it+1}}{k_{it+1}}\). Define

\[ R^k_k \left( \frac{l_{it+1}}{k_{it+1}}, \phi_{t+1} \right) = \frac{\Pi_{it+1}}{k_{it+1}} = v_{it} \left( A(\phi_{t+1})^\alpha \left( \frac{l_{it+1}}{k_{it+1}} \right)^{1-\alpha} + (1 - \delta) \phi_{t+1} \right) \]

Then, the firm’s maximization problem becomes

\[ \Gamma_{it} = \max_{\hat{k}_{it+1}, \ lev_{it+1}, \frac{l_{it+1}}{k_{it+1}}} -\hat{k}_{it+1} + \chi q_{lev_{it+1}} + EM_{t+1} r_{t+1} \left( v_{it} R^k_{t+1} - lev_{it+1} + \frac{\Gamma_{it+1}}{k_{it+1}} \right) \]

\[ q\left(\frac{l_{it+1}}{k_{it+1}}, lev_{it+1}, S_t\right) = EM_{t+1} \left[ r_{t+1} + (1 - r_{t+1}) \frac{v_{it} R^k_{t+1} + \Gamma_{it+1}}{lev_{it+1}} \right] \]

We guess and then verify that \(\Gamma_{it} = 0\)\(^{22}\). Replacing the debt price schedule and rearranging terms yields

\[ \Gamma_{it} = \max_{\hat{k}_{it+1}, \ lev_{it+1}, \frac{l_{it+1}}{k_{it+1}}} -\hat{k}_{it+1} + \chi W_t \left( \frac{l_{it+1}}{k_{it+1}} \right) + EM_{t+1} \left( R^k_{t+1} + lev_{it+1} \right) \]

\[ \hat{k}_{it+1} = R^k_{t+1} + lev_{it+1} \left( \chi - 1 \right) r_{it+1} + (\chi \theta - 1) \left( 1 - r_{it+1} \right) v_{it} R^k_{t+1} \]

By definition of the default threshold, we have \(\mathbb{E} r_{t+1} = (1 - F(\underline{v}))\). Also, note that the default threshold becomes \(\underline{v} = \frac{lev_{it+1}}{R^k_{t+1}}\). Hence

\[ \hat{k}_{t+1} = R^k_{t+1} \left( 1 + \underline{v} \left( \chi - 1 \right) \left( 1 - F(\underline{v}) \right) + (\chi \theta - 1) h(\underline{v}) \right) \]

\(^{22}\)As the firm has constant returns to scale the problem will be linear in capital and in equilibrium \(\Gamma_{it} = 0\). See Navarro (2014).
where \( h(v) = \int_{-\infty}^{v} vdF(v) \). Finally, the problem is

\[
\Gamma_{it} = \max_{\hat{k}_{it+1}, \text{lev}_{it+1}} \hat{k}_{it+1} \left( -1 - \chi \mathcal{W}_t \frac{l_{it+1}}{k_{it+1}} + \mathbb{E} M_{t+1} R^k_{t+1} J^k(\psi) \right)
\]

\[ J^k(\psi) = 1 + (\chi - 1) \psi (1 - F(\psi)) + (\chi \theta - 1) h(\psi) \]

\[ \psi = \frac{\text{lev}_{it+1}}{R^k_{t+1}} \]

First, note that the problem is linear in \( \hat{k}_{it+1} \) therefore in equilibrium we must have that

\[
1 + \chi \mathcal{W}_t \frac{l_{it+1}}{k_{it+1}} = \mathbb{E} M_{t+1} R^k_{t+1} J^k(\psi),
\]

which implies equation 11 in the main text and in turn it verifies the guess, \( \Gamma_{it} = 0 \).

Next, the first order condition with respect to leverage is

\[
\chi \mathcal{W}_t = \mathbb{E} M_{t+1} \frac{\partial R^k_{t+1}}{\partial \text{lev}_{it+1}} = 0.
\]

where

\[
R^k_{t+1} \frac{\partial J^k(\psi)}{\partial \frac{l_{it+1}}{k_{it+1}}} = R^k_{t+1} \frac{\partial \psi}{\partial \frac{l_{it+1}}{k_{it+1}}} \left( (\chi - 1) (1 - F(\psi)) - \psi (\chi - 1) f(\psi) + (\chi \theta - 1) \frac{\partial h(\psi)}{\partial \psi} \right)
\]

\[
\frac{\partial \psi}{\partial \frac{l_{it+1}}{k_{it+1}}} = -\frac{\text{lev}_{it+1}}{(R^k_{t+1})^2} \frac{\partial R^k_{t+1}}{\partial \frac{l_{it+1}}{k_{it+1}}} = -\frac{\psi^2}{\text{lev}_{it+1}} \frac{\partial R^k_{t+1}}{\partial \frac{l_{it+1}}{k_{it+1}}}
\]

\[
\frac{\partial R^k_{t+1}}{\partial \frac{l_{it+1}}{k_{it+1}}} = \psi A (1 - \alpha) \phi_{it+1}^\alpha \left( \frac{l_{it+1}}{k_{it+1}} \right)^{-\alpha}.
\]

Rearranging terms yields

\[
\chi \mathcal{W}_t = \mathbb{E} M_{t+1} \frac{\partial R^k_{t+1}}{\partial \frac{l_{it+1}}{k_{it+1}}} J^k(\psi),
\]

\[
J^k(\psi) = 1 + \psi^2 f(\psi) \chi (1 - \theta) - (1 - \chi \theta) h(\psi),
\]

which is (13) in the main text.

Finally, the first order condition with respect to leverage is

\[
\mathbb{E} M_{t+1} R^k_{t+1} \frac{\partial J^k_{t+1}}{\partial \text{lev}_{it+1}} = 0,
\]
where

\[
\frac{\partial J_{k+1}}{\partial lev_{it+1}} = \frac{\partial v}{\partial lev_{it+1}} \left( (\chi - 1) (1 - F (v)) - (\chi - 1) \varphi (v) + (\chi \theta - 1) \varphi (v) \right)
\]

\[
= \frac{1}{R_{k+1}^t} \left( (\chi - 1) (1 - F (v)) - \chi (1 - \theta) \varphi (v) \right)
\]

hence

\[
(1 - \theta) E_t [M_{t+1} \varphi (v)] = \left( \frac{\chi - 1}{\chi} \right) E_t [M_{t+1} (1 - F (v))],
\]

which is (14) in the main text.

**B  Belief revisions post-2009**

Figure 5 in the text was generated by fixing post-2009 shock realizations (at the average) and beliefs (at \( \hat{G}_{2009} \)). In this section, we show that our results are not particularly sensitive to these assumptions, which were made purely for tractability. For this purpose, we use the version of the model without debt (\( \chi = 1, lev_{Target} = 0 \)), which is computationally much more tractable. Specifically, for periods after 2009, we subject the economy to draws from \( \hat{G}_{2009} \) and beliefs are updated with each new draw. Impulse response functions are derived by averaging across sample paths. Figure 13 shows the results, along with those that emerge from the fixed beliefs formulation of this version of the model. It shows that fixing beliefs approximates the full blown simulation remarkably well.

**C  Effect of shocks 2010-2014**

Here, we subject our baseline calibrated model to the full sequence of shocks, from 2008 through 2014. Agents’ decisions in each year are a function of the appropriate estimated distribution, in line with our anticipated utility framework. For the period after 2014, we adopt the same strategy as in Figure 5, i.e. we hold fixed the shock realizations (at their average) and beliefs (at \( \hat{G}_{2014} \)). The resulting time paths are plotted in Figure 14, along with the de-trended data. The patterns implied by the model are quite close to the observed, particularly for output and employment. As before, the model overshoots in terms of the capital response, but that could be addressed by incorporating adjustment costs or other frictions.

**D  Steady State Equations**

In steady state, \( M_t = 1 \) and the intertemporal Euler equation and labor optimality conditions reduce to:

\[
1 = \beta \left( \alpha \phi_{ss}^\alpha k_{ss}^{\alpha-1} l_{ss}^{1-\alpha} + \phi_{ss} (1 - \delta) \right)
\]

(17)

\[
l_{ss}^\gamma = W_{ss} = (1 - \alpha) \phi_{ss}^\alpha k_{ss}^{\alpha} l_{ss}^{1-\alpha}.
\]

(18)

Substituting for \( l_{ss} \) from the second into the first and re-arranging yields the expression in the text.
Figure 13: **Model vs data from 2008-2014.** Solid line is the baseline model subjected to the observed sequence of shocks from 2008-2014. The red circles are US data, in deviations from their pre-crisis trends.
Figure 14: **Model vs data from 2008-2014.** Solid line is the baseline model subjected to the observed sequence of shocks from 2008-2014. The red circles are US data, in deviations from their pre-crisis trends.